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ABSTRACT

This is one of a series of 20 booklets designed for participants in an in-service course for teachers of elementary mathematics. The course, developed by the University of Illinois Arithmetic Project, is designed to be conducted by local school personnel. In addition to these booklets, a course package includes films showing mathematics being taught to classes of children, extensive discussion notes, and detailed guides for correcting written lessons. This booklet contains exercises on rules of composition, a summary of the problems in the film "Introduction to Composition," and the supplement. (MK)

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THE ARITHMETIC PROJECT COURSE FOR TEACHERS

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TOPIC: Composition (continued).

FILM Introduction to Composition, Grade 5

SUPPLEMENT: More Problems With Composition of Number Line Rules

NAME: .....

16

SE 031 116

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BOOK SIXTEEN

I.

Recall that if rules a and b are

$\square \xrightarrow{a} 4 \times \square$

$\square \xrightarrow{b} 3 \times \square$

then the composition of rule a followed by rule b is the single rule

$\square \longrightarrow 12 \times \square$

which, for convenience, we shall label rule ab :

$\square \xrightarrow{ab} 12 \times \square$

When a student suggests $\square \longrightarrow 7 \times \square$ for rule ab (a very likely wrong answer), have him pick a starting number and use rule a followed by rule b . He should then compare his final landing point with a jump from the same starting number with his rule ab .

Suppose he suggests starting at 10 . Make a jump with rule a :

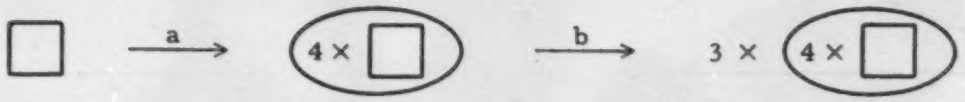
$10 \xrightarrow{a} 40$

Now, from your landing point make a jump with rule b :

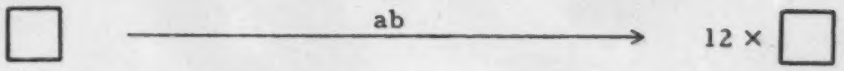
$40 \xrightarrow{b} 120$

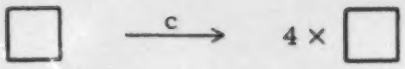
If $\square \longrightarrow 7 \times \square$ is the correct rule, it should take you to 120 when you start at 10 . Since it doesn't, it is incorrect.

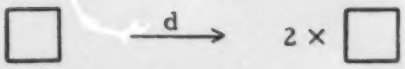
For students who have difficulty, one helpful way to consider this problem is to think of writing rule a followed by rule b in this fashion:



Thus rule ab is

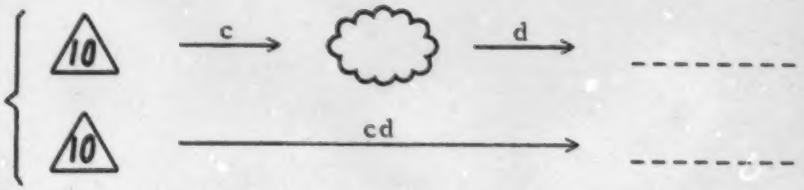


1. Rules: 

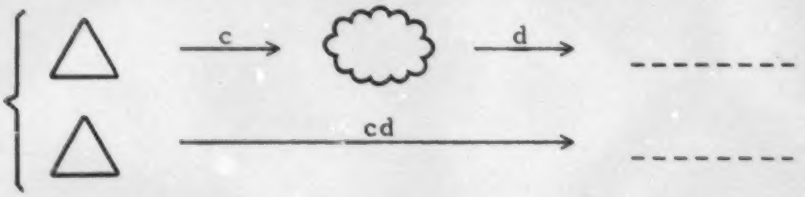


(a) Write rule cd:

Make a check starting at 10:

First check 

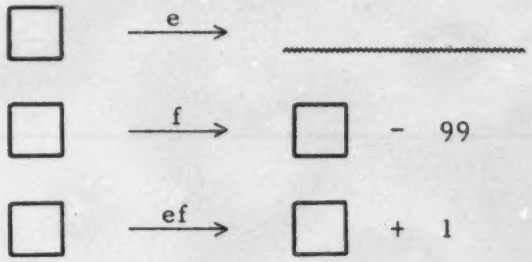
Make another check, starting at a different number:

Second check 

(b) Start at 11 and make one jump with rule cd. Land: _____

(c) Start at 5 and make two consecutive jumps with rule cd: _____

2. (a) Write rule e , given rule f and rule ef :



Compute:

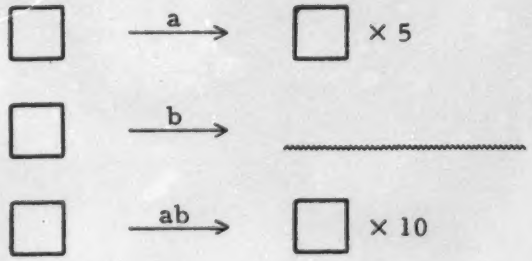
(b) 12 $\xrightarrow{e}$ = _____

(c) 12 $\xrightarrow{e}$ $\xrightarrow{f}$ = _____

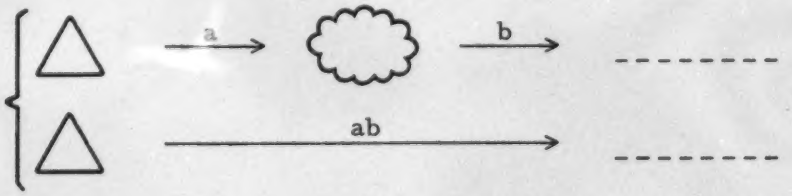
(d) 12 $\xrightarrow{ef}$ = _____

Your answers to (c) and (d) should be the same.*

3. Write rule b :



Check your rule with at least one starting point in the wedges:



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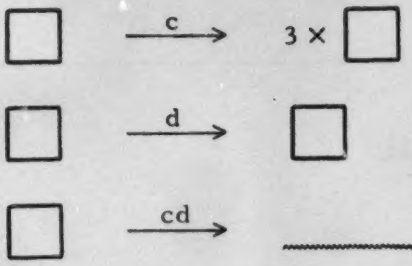
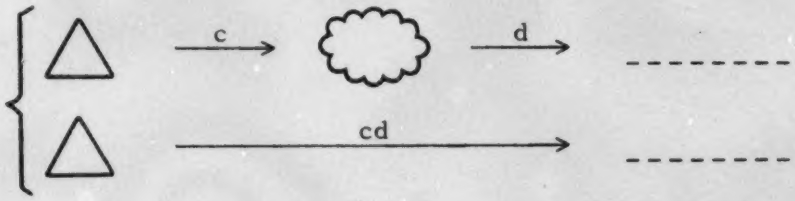
* You will note that the notation in this problem is the same as that in the fourth lesson. However, in (d) we are using "ef" as the name of the single rule □ ⟶ □ + 1 , not to indicate consecutive jumps with rule e and rule f .

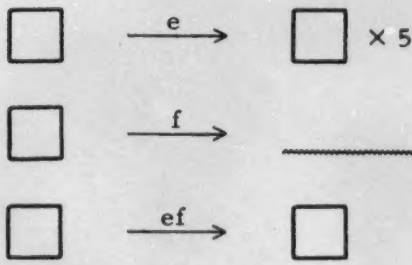
4. Write rule cd :



Pick a number for △ and see if the final landing point using rules c and d is the same as the landing point using your rule cd :

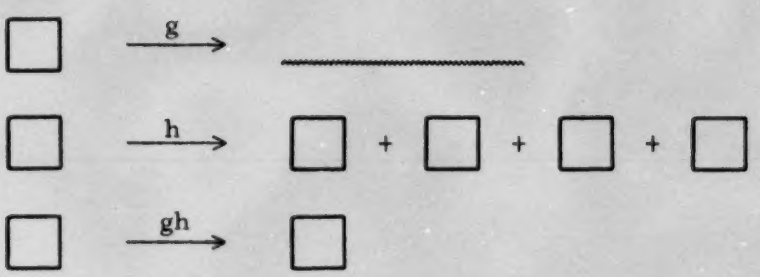


5. Write rule f :



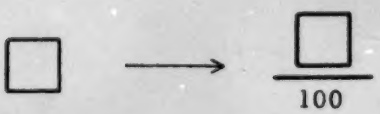
If your rule f is correct it is the <u>inverse</u> of rule e . Whatever starting point you pick, if you use rule e followed by rule f , you land back on that same number.

6. (a) Write rule g :



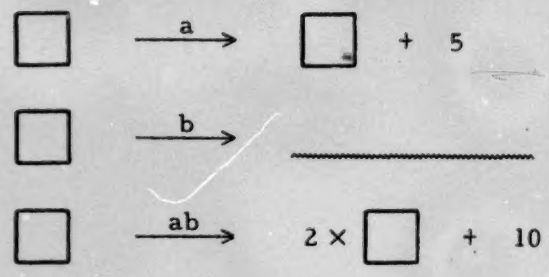
Rule g is the inverse of rule h .

(b) Write rule hg :

7. Write the inverse of 

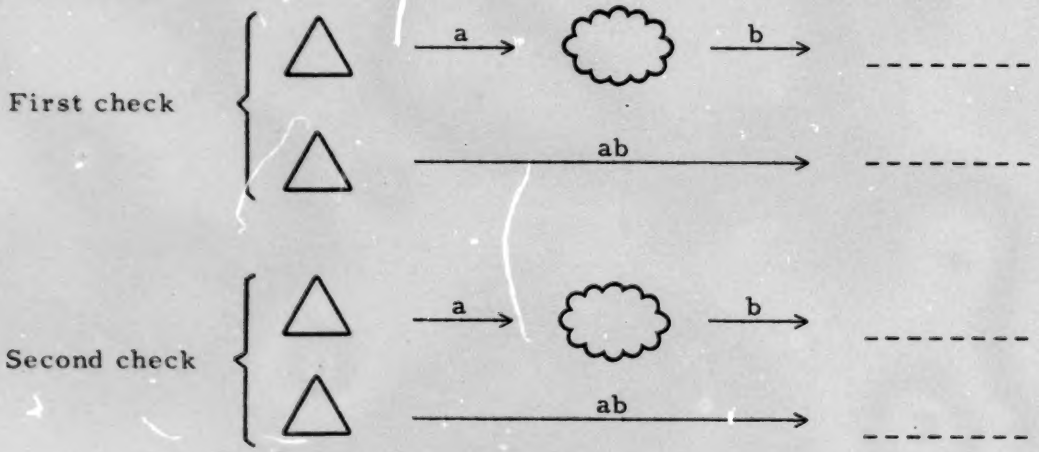
Inverse:

8. Write rule b :

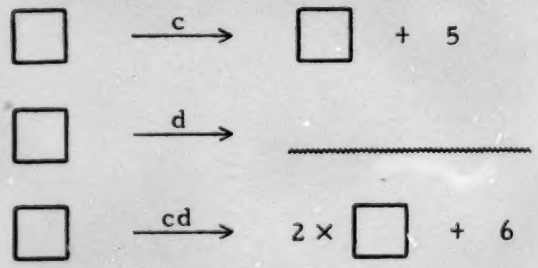


(Recall that $2 \times \square + 10$ is equivalent to $2 \times (\square + 5)$. That is, the two expressions give the same number no matter what number is put in $\square$.)

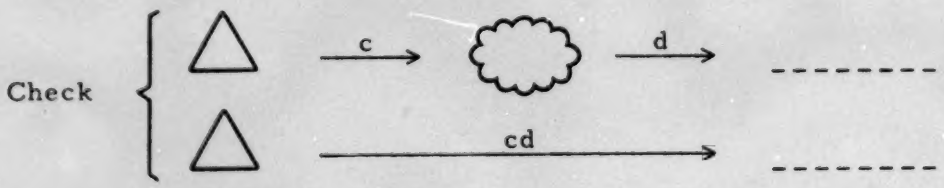
Check your rule b with two different numbers:



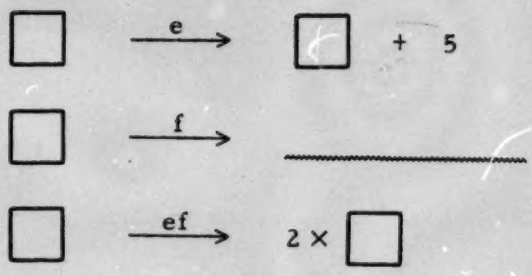
9. Write rule d :



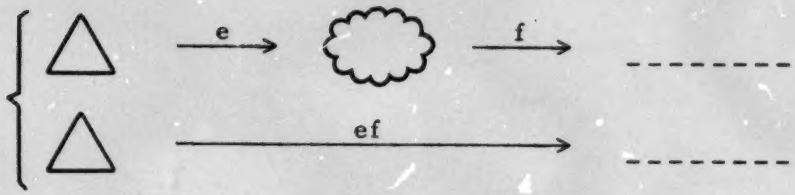
As before, check your rule with at least one number:



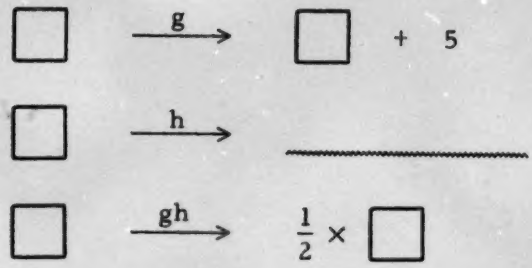
10. Write rule f :



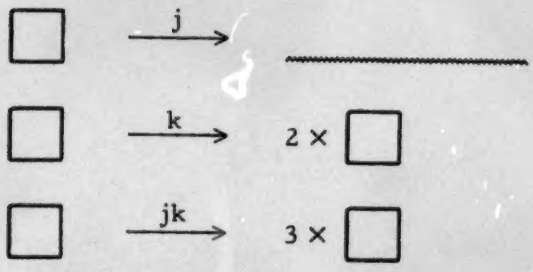
Check



11.



12.



II.

Suppose that the rules a , b , and c are

□ —a→ □ − 5

□ —b→ 3 × □

□ —c→ □ + 5

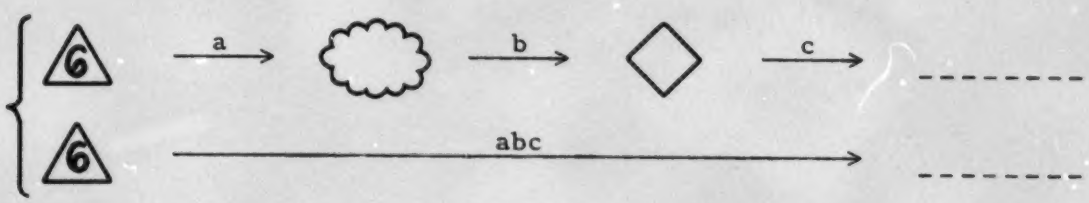
Write these rules:

1. □ —ba→ _____

2. □ —ab→ _____ (Notice that rule ab is not the same as rule ba .)

3. □ —abc→ _____

Try out your rule abc by starting at 6 :



Of course, your last two answers should be the same.

4. □ —ac→ _____

5. □ —acacacb→ _____

6. □ —bccc→ _____

7. □ —cccb→ _____

Continue with the rules: □ $\xrightarrow{a}$ □ − 5

□ $\xrightarrow{b}$ 3 × □

□ $\xrightarrow{c}$ □ + 5

8. □ $\xrightarrow{bcabb}$ _____

(Hint: The answer is not □ ⟶ 9 × □.)

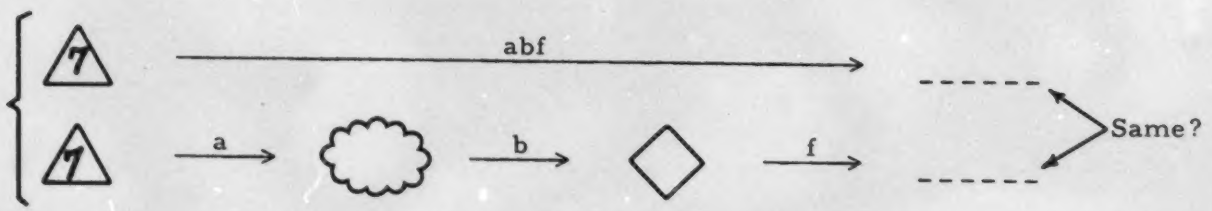
☆ 9. Write rule e so that rule abe is □ $\xrightarrow{abe}$ 3 × □

□ $\xrightarrow{e}$ _____

☆10. Write rule f :

□ $\xrightarrow{abf}$ □

□ $\xrightarrow{f}$ _____



III.

☆ 1. If rule g is □ $\xrightarrow{g}$ 10 − □, write rule h so that rule gh is □ $\xrightarrow{gh}$ □.

□ $\xrightarrow{h}$ _____

Epilogue

Composition can be a great help in planning lessons with jumping rules. For example, imagine that you want to use three rules (call them rules a , b , and c), and you need to know which combination of one a , one b , and one c will win. You could test various combinations with specific starting numbers. This might give you a feeling for which combination would win, but it is time-consuming, and in the end you still might not be certain. Using composition offers a better approach. You can write the one rule which does the same thing as using a , then b , then c . Next you can write the one rule which does the same thing as using the rules in the order c , a , b . In this manner you can write one rule for each possible combination of the letters a , b , c . Often simple inspection of the rules you obtain by composition will tell you which is the winning combination.

Here is an illustration using the following rules (answers will be found on page 12):

$\square \xrightarrow{a} 2 \times \square$

$\square \xrightarrow{b} \frac{1}{3} \times \square$

$\square \xrightarrow{c} \square + 6$

1. Write rule abc . (Hint: First find rule ab .)

$\square \xrightarrow{abc}$ _____

2. Write rule bca :

$\square \xrightarrow{bca} \underline{\left(\left(\frac{1}{3} \times \square\right) + 6\right) \times 2}$

or $\square \xrightarrow{bca} \frac{2}{3} \times \square + 12$

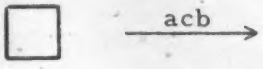
Continue with the rules: $\square \xrightarrow{a} 2 \times \square$

$\square \xrightarrow{b} \frac{1}{3} \times \square$

$\square \xrightarrow{c} \square + 6$

3. Write rule cab :

4. Write rule acb :



Which of these four rules, abc , bca , cab , or acb , gives the largest landing number for any given starting number? (Writing the rules in the form "box goes to something times box plus (or minus) something" will help you compare them.)

Using one a , one b , and one c , there are two other combinations, bac and cba .

5. Rule bac is the same as rule _____ .

6. Rule cba is the same as rule _____ .

* * *

In the film "Introduction to Composition" a student tries to state how many numbers you need to try to be sure your composed rule is correct. You may have guessed that for the linear rules we have been working with, just two numbers are necessary.

This is because the composite of any two linear rules is itself linear. Indeed, if rule a is $\square \xrightarrow{a} S \times \square + T$ and rule b is $\square \xrightarrow{b} U \times \square + V$ (where S , T , U and V are numbers), then we have

$$\square \xrightarrow{a} \boxed{S \times \square + T} \xrightarrow{b} U \times \boxed{S \times \square + T} + V$$

or $$\square \xrightarrow{ab} U \times S \times \square + U \times T + V$$

Now $U \times S$ is a number and $U \times T + V$ is a number, so rule ab is linear: it's of the form

$$\square \xrightarrow{ab} \text{some number} \times \square + \text{some number}$$

As you saw in the Epilogue of the previous written lesson, there is exactly one linear rule which takes two particular starting numbers to two given landing numbers. So if you write some linear rule as the composite of two linear rules, and if it works for two numbers, it must be right.

* * *

Answers to the problems on page 10 and page 11.

Page 10.

1. $\square \xrightarrow{abc} \frac{2}{3} \times \square + 6$

2. $\square \xrightarrow{bca} \frac{2}{3} \times \square + 12$

Page 11.

3. $\square \xrightarrow{cab} \frac{2}{3} \times \square + 4$

4. $\square \xrightarrow{acb} \frac{2}{3} \times \square + 2$

5. abc

6. cab

Summary of Problems in the Film "Introduction to Composition"

5th Grade, Phillips School, Watertown, Massachusetts
Teacher: Mrs. Marie Hermann

Note: This summary includes all the problems Mrs. Hermann gave during the class. Problems appearing here in dotted boxes are those that were not included in the film.

$8 + \square + \square = 17$ ($4\frac{1}{2}$)

$6 + \square + \square + \square = 16$

$\square + \square + \square + \square = 20$

$\square + \square + \square + \square + \square = 21$ ($4\frac{1}{5}$)

$\square + \square + \square = 19$

$3 \times (\square + 4) = 36$ (8)

$5 \times (\square + 4) = 45$

$5 \times (\square + 4) = 46$ ($5\frac{1}{5}$)

$\frac{1}{2} \times (\square + 3) = 12$

$\frac{1}{3} \times (\square + 8) = 9$

("I think I know the way ... You take the denominator, and you times it by 12 and then it is 24, and then you look at the number in parentheses, that's +3, and you take away 3 and you put 21 and add those together; it's 24 and that's half of that.")

Let's test your method:

$\frac{1}{5} \times (\square + 12) = 20$ (88)

$\square \xrightarrow{a} \square + 5$.

$\square \xrightarrow{b}$

$\square \longrightarrow 2 \times \square + 10$

I have a rule a here, and I'm not going to tell you what rule b is; this rule [$\square \longrightarrow 2 \times \square + 10$] does the same thing as using first rule a and then rule b , and I want to see if you can guess what rule b does.

(The rule $\square \longrightarrow 2 \times \square + 5$ was suggested.)

Let's try it. Start at 10; use rule a . (15)

Now use $\square \longrightarrow 2 \times \square + 5$, and tell me what you get when you are at 15.

(35)

This rule [$\square \longrightarrow 2 \times \square + 10$] sends 10 to 30, and with $\square \longrightarrow 2 \times \square + 5$, we go (from 15) to 35. Can you fix it?

("Take off the 5";
$\square \longrightarrow 2 \times \square$)

That looks like they match, but they have to match for all numbers we start with. Somebody give me a number to start with.

(9 , 400 , $9\frac{1}{4}$ were tested.)

("What would happen if you used negative numbers?")

Can you give me a negative number? (-4)

Where do you go with rule a ? (1)

And with rule b from 1 ? (2)

Now a jump from -4 with this rule here [$\square \longrightarrow 2 \times \square + 10$] which I guess I'm going to call rule ab if it seems to be doing the same thing as a and then b .

(2)

("Would $-5\frac{99}{100}$ work?")

How many think it works? Do you want to do all that work?

On board:

$$-5\frac{99}{100} \longrightarrow -\frac{99}{100} \longrightarrow -\frac{198}{100}$$

(Wrong answer suggested for $-5\frac{99}{100} + 5$: $-\frac{1}{100}$)

Now the last part.

("It's the same.")

I wrote rule ab ; could you write rule ba ? In other words, you first take a jump with rule b , then a jump with rule a , and this rule has to do the same thing.

($\square \longrightarrow 10 \times \square + 2$ was suggested.)

I guess you did that because you saw the letters go around.

("It wouldn't work.")

($\square \longrightarrow \square + (5 \times 2)$ was suggested.)

That doesn't work. Why?

("It's $\square \longrightarrow \square + 10$.")

What would you like to try now?

($\square \longrightarrow (\square + 5) \times 2$ was tested and didn't work. Then $\square \longrightarrow 10 + \square \times 2$ was suggested.)

How about that rule?

("It's the opposite of rule ab .")

What will happen if I put the same number in each of these? [$\square \xrightarrow{ab} 2 \times \square + 10$ and $\square \longrightarrow 10 + \square \times 2$]

("It will be the same.")

($\square \longrightarrow (\square + 5) + \square$ was suggested.)

Could you write that rule another way?

($\square \longrightarrow 2 \times \square + 5$)

("$\square \longrightarrow 2 \times \square + 5$, I think, works for every number.")

("I didn't test any but I know it works because ...
$2 \times 5 = 10$ and $10 + 5 = 15$
and it's the exact same way as this one——2 times box, then plus 5.")

("You only have to do three things to find out if the rule is correct. I mean you don't have to take all numbers—— well, maybe four. Take a plain number, a number with a fraction, just a plain fraction, and a negative, and if you try those you can probably tell by then if all of the numbers work.")

The class was then given the rules

$$\square \xrightarrow{cd} 3 \times \square + 21$$
$$\square \xrightarrow{dc} 3 \times \square + 7$$

and asked to figure out what the rules c and d were. Fifteen minutes were spent on this problem.

The class was given the rules

$$\square \xrightarrow{a} \square + 3$$
$$\square \xrightarrow{b} 2 \times \square$$

and quickly found rule ab and rule ba .

On board:

$\square \xrightarrow{e} \square - 7$

$\square \xrightarrow{f} 5 \times \square$

$\square \xrightarrow{fe}$ $\qquad (\square \xrightarrow{fe} 5 \times \square - 7)$

Who can give me rule ef ? $\qquad (\square \xrightarrow{ef} (\square - 7) \times 5)$

On board:

$\square \xrightarrow{g} \square + 2$

Keep that rule e and that rule f , and find rule fg .

And now rule fge ?

(A wrong answer of
$\square \longrightarrow 5 \times \square + 2 + \square - 7$
was given, then corrected to
$\square \longrightarrow 5 \times \square + 2 - 7$.)

How many think that's right?

("I got several things that can work on that ...
$\square \longrightarrow (\square - 1) \times 5$ and
$\square \longrightarrow 5 \times \square - 5$.")

Supplement

More Problems with Composition of Number Line Rules

By Edward Esty

I.

Sometimes it is interesting to start with some collection of jumping rules (one or more) and determine what jumping rules can be generated by writing all combinations of the original rules.

Let's start with a simple example. Suppose that we have only one rule at the beginning:

$$\square \xrightarrow{a} \square + 3$$

From this rule we can generate a list of rules, namely:

a
aa
aaa
aaaa
aaaaa
⋮
aaaaaaaaaaaaaaa
aaaaaaaaaaaaaaaa
⋮
(and so on, indefinitely)

List I

Rules generated by $\square \xrightarrow{a} \square + 3$

Rule aa , written out explicitly, is $\square \xrightarrow{aa} \square + 6$.

1. Write rule aaaa explicitly:*

□ $\xrightarrow{aaaa}$ _____

2. Write rule aaaaaaa :

□ $\xrightarrow{aaaaaaa}$ _____

3. Write rule $\underbrace{aaaaa \cdots aaa}_{99 \text{ letters altogether}}$. Call it rule d .

□ $\xrightarrow{d}$ _____

4. If you start at 4 and make one jump with rule aaaaa , you will land at 19 . Is there a rule on List I that will take you in one jump from 0 to 51 ?
_____ How many letters? _____

5. Is there some rule on List I such that if you start on 1,001 you land after one jump on 3,003 ? _____

(We recall that a fractional number of jumps is impossible. Either you make a jump or you don't.)

Suppose now that we had started with two rules rather than one. We will keep the same rule a :

□ $\xrightarrow{a}$ □ + 3

We will also have rule b :

□ $\xrightarrow{b}$ □ - 2

Again we can generate a list of rules, which is shown at the top of the next page:

*Answers on page 32.

a	aa	aaa	aaaa	aaaaa	aaaaaa	. . .
b	ab	aab	aaab	aaaab	aaaaab	
	ba	aba	aaba	aaaba		
	bb	abb	aabb			
.		baa	abaa			
		bab	abab			
		bba	abba			
		bbb	abbb			
			baaa			
			baab			
			bbba			
			bbbb			

List II

Rules generated by $\square \xrightarrow{a} \square + 3$ and $\square \xrightarrow{b} \square - 2$

Notice that in this list there are many rules which are represented by more than one name. For example, rules aab , aba , baa and aaaabbbb are all the same; namely, $\square \longrightarrow \square + 4$.

6. Write out rule ab explicitly:

$\square \xrightarrow{ab} \square + 1$

7. Write rule aabb explicitly:

$\square \xrightarrow{aabb}$ _____

8. Write rule aabbb :

$\square \xrightarrow{aabbb}$ _____

9. Use the fewest possible letters to name a rule from List II that will take you in one jump from 10 to 18 :

Rule: _____ (Write letters here.)

10. Use the fewest possible letters to name a rule from List II that will take you from 28 to 1 in one jump:

Rule: _____ (Write letters here.)

11. Is there a rule on List II which will take you from $483\frac{2}{3}$ to $37{,}928\frac{2}{3}$ in one jump? _____ Why or why not?

II.

Using only rules as simple as those in Section I, one can illustrate some important ideas from number theory. We first choose two rules: one which adds some positive whole number and one which subtracts some positive whole number. We next consider the list of rules generated by the two original rules, and ask this question: "Starting at 0, what numbers can be reached in one jump by using some rule in the list?"

As an example, consider the rules $\square \xrightarrow{c} \square + 20$ and $\square \xrightarrow{d} \square - 12$. Just as before, these rules generate a list of rules:

c	cc	ccc	cccc	ccccc	. . .
d	cd	ccd	cccd		
	dc	cdc			
	dd	cdd			
		dcc			
		dcd			
		ddc			
		ddd			
			dddd		

List III

Rules generated by $\square \xrightarrow{c} \square + 20$ and $\square \xrightarrow{d} \square - 12$

It's clear that we are going to be able to reach no odd number, since 0 (our starting point) is even and since 20 and 12 (the numbers added and subtracted in the rules) are both even. Can we reach every even number? If you don't see the answer immediately try different combinations to get to 6, for example. Why can't it be done? We use the same reasoning, noting this time that 0, 20 and 12 are not only multiples of 2 but also multiples of 4. Starting at a multiple of 4 and adding or subtracting multiples of 4 will give you nothing but multiples of 4 (but not necessarily <u>all</u> multiples of 4).

So the next question is: Can we reach every multiple of 4? In particular, can we get to 4 itself in one jump? The answer to the last question is yes: trial and error shows that the rule ccddd, which is $\square \xrightarrow{\text{ccddd}} \square + 4$, takes you from 0 to 4 in one jump. Once you know that you can get to 4, you know that you can get to any positive multiple of 4, for you can use the rule written with as many blocks of ccddd as you need. For instance, to get from 0 to 8 you would use the rule ccdddccddd, which is $\square \longrightarrow \square + 8$. (Of course there is a much simpler way of writing this rule, namely as cd, but that isn't the point. We're trying to show that <u>every</u> multiple of 4 can be reached.) To get from 0 to 4,004 in one jump we can use the rule:

$$\underbrace{\text{ccdddccdddccddd} \cdots \text{ccddd}}_{\text{1,001 blocks of ccddd}}$$

Why is it that 4 is the smallest reachable positive number? Many students will conjecture that it is because 4 is the largest number that "goes into" both 12 and 20. One student said, "If you make 12 and 20 into a fraction, $\frac{12}{20}$, and then divide top and bottom by the same number to get the fraction in lowest terms, the number you have to divide by is 4." Surely we can reach no positive number smaller than the greatest common divisor because each of the numbers appearing in the rules is a multiple of the greatest common divisor. We cannot prove here that in all cases the greatest common divisor actually can be reached, but the next paragraph, together with the Appendix, suggests an avenue toward a suitable proof. In any case, the fact is that the greatest common divisor of the two numbers appearing in the rules is the smallest positive number that can be reached.

Another example:

$$\square \xrightarrow{a} \square + 159$$

and $$\square \xrightarrow{b} \square - 22$$

Now 159 and 22 have no factors in common except 1; that is, 1 is the greatest common divisor of 159 and 22. (Note that $159 = 3 \times 53$ and $22 = 2 \times 11$.) We would expect, then, that some rule on the list generated by a and b would give a jump from 0 to 1, and indeed that is the case. The rule a^9b^{65} (nine a's and sixty-five b's) is $\square \longrightarrow \square + 1$. Finding this rule is not the sort of thing one does by trial and error, unless quite a bit of time is available. One systematic way to obtain such a rule is shown in the Appendix.

Some problems:

1. What is the smallest positive number that can be reached in one jump from 0, using some rule in the list generated by the rules

$$\square \xrightarrow{e} \square + 100$$

and $$\square \xrightarrow{f} \square - 25$$

Smallest positive number: _____

What combination of letters did you use? _____

2. Same question for the rules

$$\square \xrightarrow{g} \square + 27$$

and $$\square \xrightarrow{h} \square - 12$$

Smallest positive number: _____ Combination of letters: _____

3. Same question for

$$\square \xrightarrow{k} \square + 100$$

and $$\square \xrightarrow{m} \square - 101$$

Smallest positive number: _____

Write the letters if you can: _____

(If you try to use the method of the Appendix, you will find that in the first step you use one k and no m's. You must go on to the next step.)

4. Same question for

$$\square \xrightarrow{j} \square + 12$$

and $$\square \xrightarrow{n} \square + 15$$

Smallest positive number: _____ (Careful!)

Letters: _____

If one (but not both) of the rules in problem 4 had had a minus sign instead of a plus sign, the smallest positive number obtainable would have been 3. As the question stands, the answer is 12, obtained by one application of rule j — a pretty trivial problem. A more interesting problem with rules j and n as they are now is this: Consider two separate lists, one generated by rule j alone, the other generated by rule n. What is the smallest number which can be reached both by using a rule on the first list and also by using a rule on the second list?

List generated by rule j	List generated by rule n
$\square \xrightarrow{j} \square + 12$	$\square \xrightarrow{n} \square + 15$
$\square \xrightarrow{jj} \square + 24$	$\square \xrightarrow{nn} \square + 30$
$\square \xrightarrow{jjj} \square + 36$	$\square \xrightarrow{nnn} \square + 45$
⋮	⋮

We can get from 0 to 180 by using rule j^{15} and from 0 to 180 by using rule n^{12}. (This is just a paraphrase of the fact that $12 \times 15 = 15 \times 12$.) Can we reach a number smaller than 180? Yes: 60, since $0 \xrightarrow{jjjjj} 60$ and and $0 \xrightarrow{nnnn} 60$. Can we do better than that? No, because 60 is the smallest positive number divisible (with no remainder) both by 12 and by 15; that is, 60 is the smallest common multiple of 12 and 15. Students should be encouraged to reflect on the fact that the product of 12 and 15 is the same as the product of their greatest common divisor and their least common multiple. Why is this true?

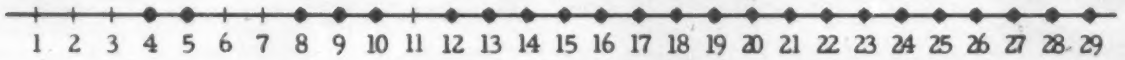
There are some questions that can be asked about the list generated by two rules each of which adds some positive integer. Consider the list generated by

$$\square \xrightarrow{f} \square + 4$$

and

$$\square \xrightarrow{g} \square + 5$$

The numbers that can be reached from 0, using some rule on the list, are shown below:



All whole numbers from 12 on can be reached. Why 12 ?

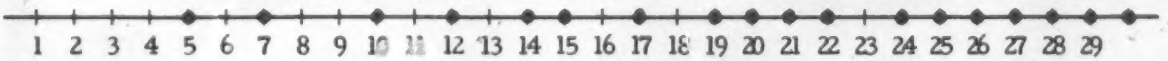
Try another. Starting at 0 and using rules from the list generated by

$$\square \xrightarrow{g} \square + 5$$

and

$$\square \xrightarrow{h} \square + 7$$

we can reach these numbers:



In this case, we can get to 24 and all whole numbers larger than 24. Why 24 ?

In general, the problem is this: Given the list generated by two rules that add positive whole numbers, what is the smallest reachable number for which all larger integers are also reachable? For an elementary school child, this is a "research problem". Suggestions:

(a) At least in the beginning choose your rules so that the numbers being added have no common factor except 1.

(b) In fact, make the numbers being added differ by 1—e.g., use $\square \longrightarrow \square + 7$ and $\square \longrightarrow \square + 8$. Keep track of your results.

(c) Then go on to differences of 2, of 3, and so on.

III.

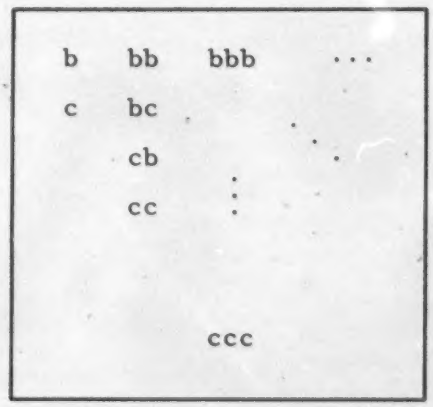
We now determine the list generated by the rules

$$\square \xrightarrow{b} \frac{1}{\square}$$

and $$\square \xrightarrow{c} -\square$$

In using the first of these rules we cannot start on 0 (why?), so we simply omit 0 as a legal starting point. (If the rule isn't familiar to you, draw a number line and make some jumps.)

Again we make our list:



List IV

List generated by $\square \xrightarrow{b} \frac{1}{\square}$ and $\square \xrightarrow{c} -\square$

What do these rules look like? We note that rule bb is

$$\square \xrightarrow{bb} \frac{1}{\frac{1}{\square}},$$

which is just $\square \longrightarrow \square$. (Try it if you don't believe this. $5 \xrightarrow{b} \frac{1}{5} \xrightarrow{b} 5$, so $5 \xrightarrow{bb} 5$; $3\frac{1}{2} \xrightarrow{b} \frac{2}{7} \xrightarrow{b} 3\frac{1}{2}$, so $3\frac{1}{2} \xrightarrow{bb} 3\frac{1}{2}$; and so on.) Rule bb, then, is $\square \xrightarrow{bb} \square$, but we will call it rule a, for short. Since rule a has no effect, rule bbb (which can be written as rule ab) is just rule b.

Rule cc is $\square \xrightarrow{cc} -(-\square)$, which is again $\square \longrightarrow \square$. (E.g., $5 \xrightarrow{c} -5 \xrightarrow{c} 5$, so $5 \xrightarrow{cc} 5$; and so on.) Thus cc is the same as a.

What about bc? We have $\square \xrightarrow{bc} -\frac{1}{\square}$. (E.g., $5 \xrightarrow{b} \frac{1}{5} \xrightarrow{c} -\frac{1}{5}$, so $5 \xrightarrow{bc} -\frac{1}{5}$.) We rename rule bc and call it rule d.

And what about cb? This turns out to be the same as bc, for $\square \xrightarrow{cb} \frac{1}{-\square}$ is the same as $\square \xrightarrow{bc} -\frac{1}{\square}$. So rule cb is the same as rule d.

Thus far we have four rules on our list:

$$\square \xrightarrow{a} \square$$

$$\square \xrightarrow{b} \frac{1}{\square}$$

$$\square \xrightarrow{c} -\square$$

$$\square \xrightarrow{d} -\frac{1}{\square}$$

Are there other rules on the list? Surprisingly enough, no. Consider any string of b's and c's, for example bccbbccc. Using the fact that bc = cb, we can switch the letters around so that all the b's appear at the beginning:

b c <u>c b</u> b c c c

b <u>c b</u> c b c c c

b b c <u>c b</u> c c c

b b <u>c b</u> c c c c

b b b c c c c c

At each step, the cb pair to be reversed has been underlined.

Now whenever two b's appear together we can cross them out, for rule bb is $\square \xrightarrow{a} \square$, which has no effect. Similarly we can cross out pairs of c's. In our example we would have:

b b b c c c c c

~~b b~~ b c ~~c c c c~~

b c , or d

In general, no matter what string of b's and c's appear, we can proceed as in our example until we arrive at one of the following situations:

(i) Nothing is left at all. The original rule was the same as rule a , $\square \longrightarrow \square$.

(ii) One b is left. The original rule was rule b .

(iii) One c is left. The original rule was rule c .

(iv) One b and one c are left. The original rule was bc , or d .

Thus every combination of b's and c's can be collapsed into a , b , c or d ; there is so much repetition on our infinite list of strings of b's and c's that in the entire list only four different rules appear.

Clearly, then, if we take any two of these four rules and combine them by composition we will get a rule that's already one of the four. Indeed, we can make a table showing each possible pair of rules and the rule which is their composite:

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

Compare this table with the one shown on page 11 of the written lesson for Book 11. The two tables are identical. Thus our four rules, a , b , c , and d , together with the binary operation of composition, form a concrete example of what was originally introduced as a completely abstract system.

IV. A natural question now is this: Is there a collection of rules such that the list generated by them again contains only four different rules but which gives a table exactly like the one on the bottom of page 10 of the written lesson for Book 11 ? The table there was

	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

The answer is yes. As a matter of fact, the list can be generated by the single rule

$$\square \xrightarrow{b} (0 \vee \square) - \left(\frac{0 \vee \square}{\square \times \square}\right) - \square$$

where $0 \vee \square$ means $\max(0, \square)$. As before, we exclude 0 as a starting point to avoid dividing by zero. Make some jumps with rule b . For example,

$$5 \xrightarrow{b} (0 \vee 5) - \left(\frac{0 \vee 5}{5 \times 5}\right) - 5$$

$$5 \xrightarrow{b} 5 - \frac{5}{25} - 5$$

$$5 \xrightarrow{b} -\frac{1}{5}$$

Make another jump:

$$-\frac{1}{5} \xrightarrow{b} (0 \vee -\frac{1}{5}) - \left(\frac{0 \vee -\frac{1}{5}}{-\frac{1}{5} \times -\frac{1}{5}}\right) - (-\frac{1}{5})$$

$$-\frac{1}{5} \xrightarrow{b} 0 - 0 + \frac{1}{5}$$

$$-\frac{1}{5} \xrightarrow{b} \frac{1}{5}$$

So 5 $\xrightarrow{bb}$ $\frac{1}{5}$. Make another jump from $\frac{1}{5}$. Where do you land? Make still another. The reader can verify that making four jumps with rule b always gets you back to where you started. Rule bbbb is therefore just $\square \longrightarrow \square$. Given any string of b's we can cross off blocks of four, until we are left with no b's ($\square \longrightarrow \square$), one (rule b), two (rule bb), or three (bbb). If we relabel rule bb as c, let d = bbb and let a = bbbb (so that rule a is $\square \xrightarrow{a} \square$, as before), then the table showing all possible ways of composing these four rules will look exactly like the one on the preceding page.

V. Two further problems with lists:

1. The rules

$$\square \xrightarrow{g} \frac{1}{\square}$$

and

$$\square \xrightarrow{h} 1 - \square$$

generate only a finite number of different rules, although this time there are more than four of them. How many are there, what are they, and what does the composition table look like? (Again 0 is excluded as a starting number. 1 must be excluded too. Why?)

2. The rules

$$\square \xrightarrow{p} \square + 3$$

and

$$\square \xrightarrow{q} \frac{\square}{2}$$

generate an infinite number of different rules. Notice that there is some repetition on the list, though, since rule ppq and rule qp are both

$$\square \longrightarrow \frac{\square}{2} + 3$$

Is there a rule on the list that will give a jump from 0 to 1? If not, how close can you get to 1?

Appendix

We are trying to find a rule on the list generated by

$$\square \xrightarrow{a} \square + 159$$

and $$\square \xrightarrow{b} \square - 22$$

which will give a jump from 0 to 1.

First find the rule that gives jumps to the right and which has one a and as many b's as possible. We find that we can use 7 b's:

$$\square \xrightarrow{abbbbbbb} \square + 5$$

For simplicity, call this rule c.

Now find the rule that gives jumps to the left and uses one b and as many c's as possible. We find that we can use four c's:

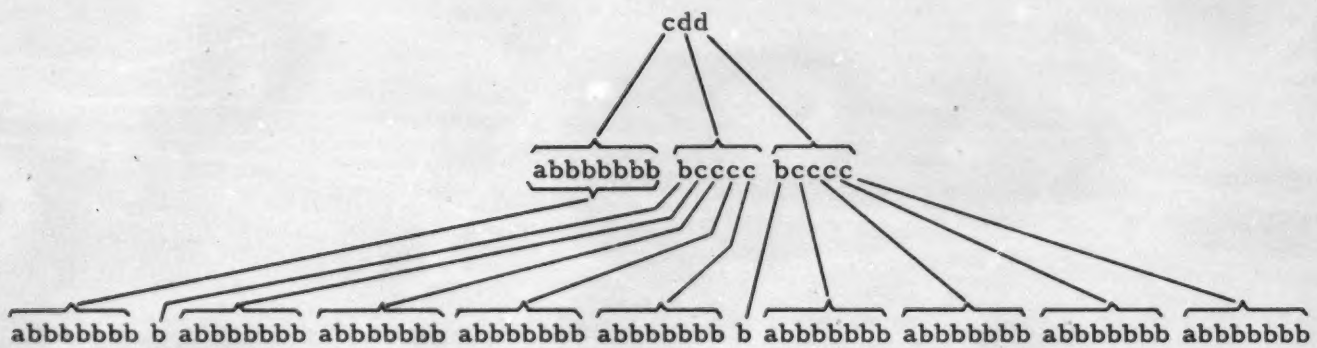
$$\square \xrightarrow{bcccc} \square - 2$$

Call this rule d.

Now find the rule that gives jumps to the right and uses one c and as many d's as possible. We can use two d's:

$$\square \xrightarrow{cdd} \square + 1$$

Translating back into a's and b's, we have



which is the same as a^9b^{65}, since the a's and b's can be put in any order.

Answers

I. 1. $\square \longrightarrow \square + 12$

2. $\square \longrightarrow \square + 21$

3. $\square \longrightarrow \square + 297$

4. Yes. 17 letters.

5. No.

7. $\square \longrightarrow \square + 2$

8. $\square \longrightarrow \square$

9. aaaabb (in any order)

10. ab^{15}

11. Yes. $37{,}928\frac{2}{3} - 483\frac{2}{3}$ is a whole number. Use the rule with enough ab pairs to get you from $483\frac{2}{3}$ to $37{,}928\frac{2}{3}$.

II. 1. 25 ; efff

2. 3 ; ghh

3. 1 ; $k^{100}m^{99}$

4. 12 ; j

V. 1. There are six rules:

$\square \xrightarrow{g} \frac{1}{\square}$

$\square \xrightarrow{h} 1 - \square$

$\square \xrightarrow{gg} \square$

$\square \xrightarrow{gh} 1 - \frac{1}{\square}$

$\square \xrightarrow{hg} \frac{1}{1 - \square}$

$\square \xrightarrow{hgh} 1 - \frac{1}{1 - \square}$

Letting gg = a , gh = b , hg = c , and hgh = d , the table is the one shown at the right.

	a	b	c	d	g	h
a	a	b	c	d	g	h
b	b	c	a	h	d	g
c	c	a	b	g	h	d
d	d	g	h	a	b	c
g	g	h	d	c	a	b
h	h	d	g	b	c	a

2. No rule on the list gives a jump from 0 to 1 , but you can get as close to 1 as you please.